• الاسبوع ٣ رياضة

• Sec 2 (م/ تحويلات)

1 * Fourier Series:-

* اى دالة يمكن تمثيلها بمجموع Sin, Cos

بشرط ⟵ الدالة تكون, Piecewise continous

⟵ Bounded

⟵ Periodic

* $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$

$\longrightarrow$ if P (period) $= 2\pi$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos(nx)\, dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin(nx)\, dx$$

$\longrightarrow$ if P (period) $= 2L$

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\, dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

→ even function

$f(-x) = f(x)$

$\int_{-L}^{L} \text{even} = 2\int_{0}^{L} \text{even}$

* odd function

$f(-x) = -f(x)$

$\int_{-L}^{L} \text{odd} = 0$

* Rules:-

even * even = even

even * odd = odd

odd * odd = even

$\sin A \cos B = \frac{1}{2}\left[\sin(A-B) + \sin(A+B)\right]$

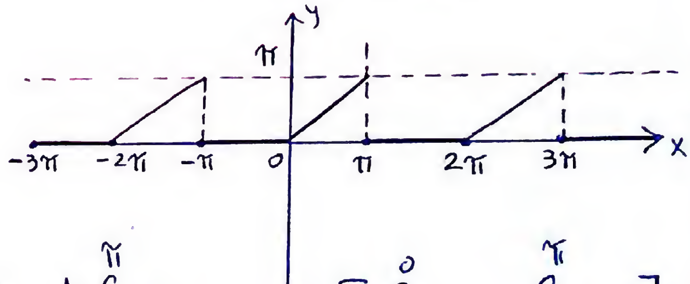
$\cos A \cos B = \frac{1}{2}\left[\cos(A-B) + \cos(A+B)\right]$

$\sin A \sin B = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right]$

* Sheet 1 :-

→ find Fourier Series for:-

① $f(x) = \begin{cases} 0 & -\pi \le x < 0 \\ x & 0 \le x \le \pi \end{cases}$, $P = 2\pi$



$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\, dx + \int_{0}^{\pi} x\, dx\right]$$

$$= \frac{1}{\pi} * \frac{x^2}{2}\Big|_0^{\pi} = \pi/2 .$$

$$a_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} 0 . \cos x\, dx + \int_{0}^{\pi} x \cos x\, dx\right]$$

$$= \frac{1}{\pi}\left(\underbrace{\frac{x \sin nx}{n}}_{=0} + \frac{\cos nx}{n^2}\right)\Big|_0^{\pi}$$

$$= \frac{1}{\pi}\left(\frac{\cos n\pi}{n^2} - \frac{1}{n^2}\right)$$

$$= \frac{1}{\pi}\left(\frac{(-1)^n - 1}{n^2}\right) = \frac{(-1)^n - 1}{\pi n^2}$$

* $\cos(n\pi) = (-1)^n$, $\sin(n\pi) = 0$

* $\cos(2n\pi) = 1$, $\cos(2n-1)\pi = -1$

$$b_n = \frac{1}{\pi}\int_0^{\pi} x \sin nx \, dx$$

$$= \frac{1}{\pi}\left(\frac{-x\cos nx}{n} + \underbrace{\frac{\sin nx}{n^2}}_{=0}\right)\Bigg|_0^{\pi}$$

$$= \frac{1}{\pi}\left(\frac{-\pi(-1)^n}{n} + 0 + 0\right) = \frac{(-1)^{n+1}}{n}$$

$$\Rightarrow \therefore f(x) = \frac{\pi}{4} + \sum_{n=1}^{\infty}\left(\frac{(-1)^n - 1}{\pi n^2}\cos nx + \frac{(-1)^{n+1}}{n}\sin nx\right).$$

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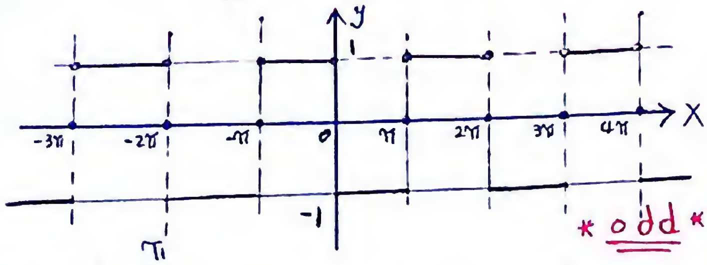
② $f(x) = \begin{cases} 1 & -\pi < x < 0 \\ -1 & 0 \le x < \pi \end{cases}$



* odd *

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{f(x)}_{odd}\, dx = 0.$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cdot \cos nx \, dx = 0$$

odd * even = odd

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cdot \sin nx \, dx$$

odd * odd = even

$$= \frac{+2}{\pi}\int_0^{\pi} -\sin nx \, dx = \frac{2}{\pi}\,\frac{\cos nx}{n}\Bigg|_0^{\pi}$$

$$= \frac{2}{\pi n}\left((-1)^n - 1\right).$$

$$\Rightarrow \therefore f(x) = \sum_{n=1}^{\infty}\left(\frac{2}{\pi n}\left((-1)^n - 1\right)\sin nx\right).$$

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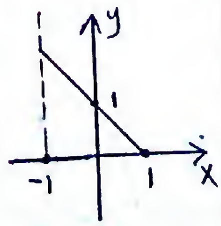
③ $f(x) = 1 - x \quad , -1 \le x \le 1 \quad (P = 2)$

$$a_0 = \frac{1}{1}\int_{-1}^{1}(1-x)\,dx$$

$$= x - \frac{1}{2}x^2\Big|_{-1}^{1} = 2$$



$$a_n = \frac{1}{1}\int_{-1}^{1}(1-x)\cos n\pi x \, dx$$

$$= \int_{-1}^{1}\cos(n\pi x)\,dx - \int_{-1}^{1} x\cos(n\pi x)\,dx$$

$$a_n = \underbrace{\frac{-\sin(n\pi x)}{n\pi}}_{=0}\Bigg|_{-1}^{1} - \left(\underbrace{\frac{x\sin n\pi x}{n\pi}}_{=0} + \frac{\cos n\pi x}{n^2\pi^2}\right)\Bigg|_{-1}^{1}$$

$$\cancel{\frac{2(-1)^n}{n^2\pi^2}} = 0$$

$$b_n = \frac{1}{1}\int_{-1}^{1}(1-x)\sin n\pi x \, dx$$

$$= \int_{-1}^{1}\sin n\pi x \, dx - \int_{-1}^{1} x\sin n\pi x \, dx$$

$$= \underbrace{-\frac{\cos n\pi x}{n\pi}\Bigg|_{-1}^{1}}_{=0} - \left(-x\frac{\cos n\pi x}{n\pi} + \underbrace{\frac{\sin n\pi x}{n^2\pi^2}}_{=0}\right)\Bigg|_{-1}^{1}$$

$$= x\,\frac{\cos n\pi x}{n\pi}\Bigg|_{-1}^{1} = \frac{2(-1)^n}{n^2\pi^2}.$$

$$\Rightarrow \therefore f(x) = 1 + \sum_{n=1}^{\infty}\left(\frac{2(-1)^n}{n^2\pi^2}\sin n\pi x\right).$$

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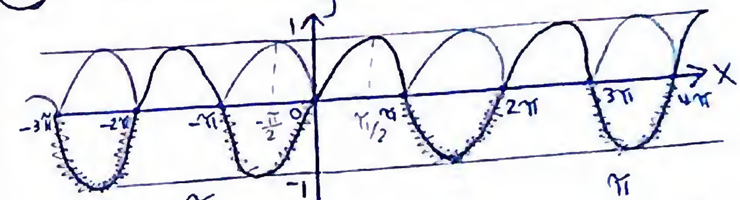
④ $f(x) = \sin x \quad , 0 \le x \le \pi \quad , P = \pi$



$$a_0 = \frac{1}{\pi/2}\int_0^{\pi}\sin x \, dx = \frac{2}{\pi} * -\cos x\Big|_0^{\pi} = \frac{4}{\pi}$$

hint

لا يجوز حل التكامل بالحدود $\int_{-\pi/2}^{\pi/2}$ لأن فترة الدالة من $\pi \to 0$
وحل التكامل بهذه الطريقة (يأخذ جزء من الدالة وليس كلها)

$$a_n = \frac{2}{\pi}\int_0^{\pi}\sin x \cos 2nx \, dx \qquad \left(\frac{n\pi x}{L} = \frac{n\pi x}{\pi/2} = 2nx\right)$$

$$= \frac{2}{\pi}\int_0^{\pi}\frac{1}{2}\sin(1-2n)x + \sin(1+2n)x \, dx$$

$$= \frac{1}{\pi}\left(\frac{-\cos(1-2n)x}{(1-2n)} - \frac{\cos(1+2n)x}{1+2n}\right)\Bigg|_0^{\pi}$$

$$= \frac{1}{\pi}\left(\frac{2}{1-2n} + \frac{2}{1+2n}\right)$$

$$= \frac{1}{\pi}\left(\frac{2 + \cancel{4n} + 2 - \cancel{4n}}{1-4n^2}\right)$$

$$= \frac{1}{\pi}\left(\frac{4}{1-4n^2}\right) = \frac{4}{\pi(1-4n^2)}.$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} \sin x \cdot \sin 2nx \, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} \frac{1}{2}\left(\cos(1-2n)x - \cos(1+2n)x\right) dx$$

$$= \frac{1}{\pi}\left[\frac{\sin(1-2n)x}{\underbrace{1-2n}_{=0}} - \frac{\sin(1+2n)x}{\underbrace{1+2n}_{=0}}\right]\Bigg|_0^{\pi}$$

$$= 0$$

$$\therefore f(x) = \frac{2}{\pi} + \sum_{n=1}^{\infty}\left[\frac{4}{\pi(1-4n^2)} \cos n\pi x\right].$$

(5) expand $f(x) = x^2$, $0 \le x \le 1$ half-Period ($L=1$)

→ in case of :-

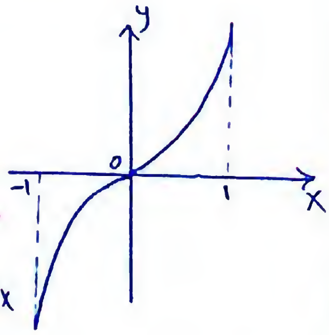
1) Sine Series only.

2) Cosine Series only → (H.W)

→ Sine Series ←

→ it should be odd.

$$a_0 = a_n = 0$$



$$b_n = \frac{1}{L}\int_{-L}^{L} \underset{\text{odd}}{f(x)} * \underset{\text{odd}}{\sin \frac{n\pi x}{L}} \, dx \quad = \text{even}$$

$$= 2\int_0^1 x^2 \sin n\pi x \, dx$$

$$= 2\left[-x^2\frac{\cos n\pi x}{n\pi} + 2x\frac{\cancel{\sin n\pi x}}{\underbrace{n^2\pi^2}_{=0}} + \frac{2\cos n\pi x}{n^3\pi^3}\right]\Bigg|_0^1$$

$$= 2\left[\frac{-(-1)^n}{n\pi} + \frac{2(-1)^n}{n^3\pi^3} - \left[\frac{2}{n^3\pi^3}\right]\right]$$

$$= 2\left(\frac{(-1)^{n+1}}{n\pi} + \frac{2[(-1)^n - 1]}{n^3\pi^3}\right)$$

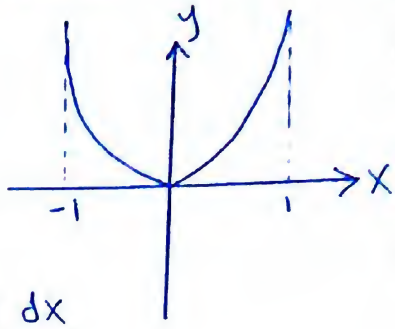
$$= \frac{2(-1)^{n+1}}{n\pi} + \frac{4((-1)^n - 1)}{n^3\pi^3}.$$

$$\therefore f(x) = \sum_{n=1}^{\infty}\left[\left(\frac{2(-1)^{n+1}}{n\pi} + \frac{4((-1)^n-1)}{n^3\pi^3}\right)\cancel{\sin \pi x}\ \sin n\pi x\right]$$

→ Cosine Series ←

→ it should be even.

$$b_n = 0$$



$$a_0 = \frac{2}{1}\int_0^1 x^2 \, dx$$

$$= 2\,\frac{x^3}{3}\Bigg|_0^1 = \frac{2}{3}.$$

$$a_n = \frac{1}{1}\int_{-1}^{1} x^2 \cos n\pi x \, dx$$

$$= 2\int_0^1 x^2 \cos n\pi x \, dx$$

$$= 2\left[x^2\frac{\cancel{\sin n\pi x}}{\underbrace{n\pi}_{=0}} + 2x\frac{\cos n\pi x}{n^2\pi^2} - 2\frac{\cancel{\sin n\pi x}}{\underbrace{n^3\pi^3}_{=0}}\right]\Bigg|_0^1$$

$+x^2$	$\cos n\pi x$
$-2x$	$\frac{\sin n\pi x}{n\pi}$
$+2$	$-\frac{\cos n\pi x}{n^2\pi^2}$
0	$-\frac{\sin n\pi x}{n^3\pi^3}$

$$= 2\left(2x\frac{\cos n\pi x}{n^2\pi^2}\right)\Bigg|_0^1$$

$$= 2 * 2\,\frac{(-1)^n}{n^2\pi^2} = \frac{4(-1)^n}{n^2\pi^2}.$$

$$\therefore f(x) = \frac{1}{3} + \sum_{n=1}^{\infty}\left(\frac{4(-1)^n}{n^2\pi^2}\cos n\pi x\right).$$